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Bending of electron edge states in a magnetic field

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Abstract. We discuss a two-dimensional electron gas cut along the half-axis $x > 0$ by an infinitely high potential barrier and calculate the probability for an electron edge state to bend around the edge of the cut. The problem is solved exactly by the Wiener–Hopf technique.

1. Introduction

A number of properties of the two-dimensional electron gas (2DEG) in a strong magnetic field are related to electron edge states [1, 2]. The most prominent property is the quantum Hall effect (QHE) [3–5], especially the anomalous QHE, when the edge states are non-equilibrially populated [6–9]. The edge states are very useful in analysing the magnetoresistance of ballistic microstructures (quantum channels and quantum dots), where one observes quenching and other anomalous properties of the QHE [10–12]. Coherent electron focusing in the 2DEG [13] is also related to edge states.

Edge-state scattering plays a dominant role in all the above-mentioned effects. There are several types of scattering of these states: scattering by impurities, phonons and irregularities of the 2DEG boundary, bending around the boundary corners, and penetration through constrictions (quantum point contacts). Scattering of edge states by impurities and phonons is discussed in [14, 15], scattering by irregularities of the boundary is discussed in [16, 17], and penetration through small constrictions is discussed in [17]. In this paper we address to the problem of edge-state bending around a corner. We have succeeded in finding the exact quantum solution of the problem in the case when the angle at the corner is 2π.

Consider an electron ($e < 0$) in plane (x, y) with magnetic field H in the z direction. The plane is cut along the half axis $x > 0$ by an infinitely high potential barrier (figure 1). In the far right side of the upper half-plane ($y > 0, x = +\infty$) there exist edge states Ψ_{El}, propagating from right to left, while in the far right side of the lower half-plane ($y < 0, x = +\infty$) there are edge states $\tilde{\Psi}_{El}$, propagating from left to right. Here E is the electron energy and $l = 0, 1, 2, \ldots$ is the Landau level number, which labels the different branches of the edge states.

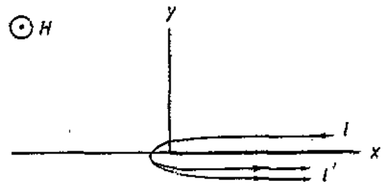

Figure 1. The xy plane. The heavy line ($x > 0$) is the potential barrier. The incoming edge state l is transformed after bending into a superposition of two edge states l'.

Let some edge wave Ψ_{El} come from $x = +\infty$ in the upper half-plane. Then the field ψ in the lower half-plane at $x = +\infty$ is a superposition of edge waves $\bar{\Psi}_{El'}$ with the same energy

$$\psi = \sum_{l'} S_{ll'}(E)\bar{\Psi}_{El'}. \tag{1}$$

The coefficients $S_{ll'}(E)$ generate the scattering matrix of the bending diffraction problem. The aim of the present paper is to calculate this matrix.

The way of solving the diffraction problem is as follows. First we derive the integral equation for the field ψ at the continuation of the cut ($y = 0$, $x < 0$). This equation is solved by the Wiener–Hopf technique [18]. Next, the scattering matrix can be calculated. Unexpectedly, the squares $|S_{ll'}(E)|^2$ are given as ratios of simple polynomials in the edge-state wavevectors $k_l(E)$.

2. Integral equation

The diffraction problem formulated in the introduction is to solve the Schrödinger equation

$$\mathcal{H}\psi = E\psi$$
$$\mathcal{H} = \frac{1}{2m}\left[\left(-\mathrm{i}\frac{\partial}{\partial x} - \frac{|e|H}{c}y\right)^2 + \left(-\mathrm{i}\frac{\partial}{\partial y}\right)^2\right] \tag{2}$$

with the following boundary conditions:

(i) $\psi = 0$ at $y = \pm 0$ for $x > 0$;
(ii) $\psi = 0$ at $y = \pm\infty$ for all x values;
(iii) $\psi = \Psi_{El}$ at $x = +\infty$ for $y > 0$;
(iv) ψ is a linear combination of edge states $\bar{\Psi}_{El}$ with the same energy E at $x = +\infty$ for $y < 0$; and
(v) $\psi = 0$ at $x = -\infty$ for all y values.

Define the 'scattered' field ψ' by

$$\psi = \psi^0 + \psi' \tag{3}$$

where ψ^0 is the 'incoming' field, i.e. $\psi^0 = \Psi_{El}$ for $y > 0$ and $\psi^0 = 0$ for $y < 0$. Field ψ' satisfies the same equation and the same boundary conditions as does field ψ, with the only exception that instead of (iii) the boundary condition is $\psi' = 0$ at $x = +\infty$, $y > 0$.

The integral equation for the field $\psi'(x<0)=\psi(x<0)\equiv\varphi(x)$ at the half-axis $x<0$ can be deduced, using the Green theorem from [17]:

$$\psi'(r)=\frac{1}{2m}\oint_C \mathrm{d}l' \left(\psi'(r')\frac{\partial}{\partial n'}G(r,r')-G(r,r')\frac{\partial}{\partial n'}\psi'(r')\right) - \frac{ie}{mc}\oint_C \mathrm{d}l'\ \boldsymbol{n}'A(r')\psi(r')G(r,r'). \tag{4}$$

Here C is a closed contour, the point r being inside, n' is the external normal and the integration over $\mathrm{d}l'$ goes anticlockwise. The Green function G obeys

$$[\mathcal{H}(r)-E]G(r,r')=-\delta(r-r') \tag{5}$$

with the Hamiltonian from (1).

Let C be a contour going first along the x axis from $x=-\infty$ to $x=+\infty$ at $y=+0$ and then along an infinitely distant semicircle at $y>0$. Let the Green function in (5) be the outgoing-wave Green function of the upper half-plane, defined in the appendix. Taking into account the gauge (A1), the symmetry property (A14) and the boundary condition (i) for ψ', we obtain

$$y>0: \qquad \psi'=-\frac{1}{2m}\int_{-\infty}^{0}\mathrm{d}x'\ \varphi(x')\left[\frac{\partial}{\partial y'}G(x,y;x',y')\right]_{y'=0}. \tag{6}$$

The integral over the distant semicircle vanishes as usual, owing to infinitesimal dissipation.

When C is a similar contour in the lower half-plane we obtain

$$y<0: \qquad \psi'=\frac{1}{2m}\int_{-\infty}^{0}\mathrm{d}x'\ \varphi(x')\left[\frac{\partial}{\partial y'}\tilde{G}(x,y;x',y')\right]_{y'=0} \tag{7}$$

where $\tilde{G}$ is the Green function of the lower half-plane. From symmetry arguments it follows that

$$\tilde{G}(x,y;x',y')=G(-x,-y;-x',-y'). \tag{8}$$

The desired equation for φ can be obtained from the continuity of the derivative $\partial\psi/\partial y$ at $y=0, x<0$, i.e.

$$\left.\frac{\partial\psi^0}{\partial y}\right|_{y=+0}+\left.\frac{\partial\psi'}{\partial y}\right|_{y=+0}=\left.\frac{\partial\psi'}{\partial y}\right|_{y=-0}. \tag{9}$$

Calculation of the derivative $\partial\psi'/\partial y$ on the LHS of (9) from (6) yields

$$\left.\frac{\partial}{\partial y}\psi'(x,y)\right|_{y=+0}=\int_{-\infty}^{0}\mathrm{d}x'\ \varphi(x')K(x-x') \tag{10}$$

with

$$K(x-x')=-\frac{1}{2m}\left[\frac{\partial^2}{\partial y\partial y'}G(x,-x';y,y')\right]_{y=y'=0}. \tag{11}$$

Making use of (8) one can calculate the derivative $\partial\psi'/\partial y$ on the RHS of (9) from (7), giving

$$\left.\frac{\partial}{\partial y}\psi'(x,y)\right|_{y=-0}=-\int_{-\infty}^{0}\mathrm{d}x'\ \varphi(x')K(x-x'). \tag{12}$$

Now introduce (10) and (12) into (9) and assume the field ψ^0 to be an edge state normalized to unit flux (see appendix). As a result, we obtain from (9) the integral equation for $\varphi(x)$ at $x<0$:

$$\int_{-\infty}^{0} \mathrm{d}x'\, \varphi(x')[K(x-x')+K(x'-x)] = -\frac{(2m)^{1/2}}{a_{\mathrm{H}}}\mathrm{e}^{ik_f x}. \tag{13}$$

To calculate the kernel of this equation in an explicit form we use the representation (A12) of the Green function. It follows from (A14) that

$$K(x) = -\frac{1}{2m}\int \frac{\mathrm{d}k}{2\pi}\mathrm{e}^{ikx}\Psi_k'(0)\Phi_k'(0). \tag{14}$$

Put $y=0$ in (A17) and use (A18). Then the derivatives are given by

$$\begin{aligned}\Psi_k'(0) &= \frac{\sqrt{2}}{a_{\mathrm{H}}}AD_\nu'(-\xi)\\ \Phi_k'(0) &= -\frac{\sqrt{2}}{a_{\mathrm{H}}}B\frac{(2\pi)^{1/2}}{\Gamma(-\nu)}.\end{aligned} \tag{15}$$

Substituting these derivatives into (14) and making use of (A19), one obtains

$$K(x) = \frac{\sqrt{2}}{a_{\mathrm{H}}}\int\frac{\mathrm{d}k}{2\pi}\mathrm{e}^{ikx}\frac{D_\nu'(-\xi)}{D_\nu(-\xi)}. \tag{16}$$

It is convenient to use the non-dimensional variable $t=-x/\sqrt{2}\,a_{\mathrm{H}}$ and the non-dimensional field $v=\varphi/m^{1/2}$. In terms of these variables (13) takes the following form:

$$\int_0^\infty \mathrm{d}t'\, v(t')\,\mathscr{L}(t-t') = f(t) \qquad t>0 \tag{17}$$

where†

$$\begin{aligned} f(t) &= -\mathrm{e}^{-\mathrm{i}\xi_f t}\\ \mathscr{L}(t) &= \int\frac{\partial\zeta}{2\pi}\frac{D_\nu'(-\zeta)}{D_\nu(-\zeta)}(\mathrm{e}^{\mathrm{i}\zeta t}+\mathrm{e}^{-\mathrm{i}\zeta t}) = \mathscr{L}(-t).\end{aligned} \tag{18}$$

The integration path in the complex plane $\zeta=\xi+\mathrm{i}\eta$ goes near the real axis from $\zeta=-\infty$ to $\zeta=+\infty$ and encompasses the real poles from above (see appendix).

Before solving equation (17), let us investigate the behaviour of the kernel $\mathscr{L}(t)$ as $t\to\pm\infty$. Since $\mathscr{L}(t)$ is symmetric, assume $t>0$.

In the first term of the integral (18) the integration path can be closed, adding a semicircular contour at infinity in the upper half-plane $\eta>0$. Only complex poles at ζ_S contribute to the integral, and as a result the first term vanishes exponentially as $t\to+\infty$. In the second term of the integral (18) a semicircle in the lower half-plane $\eta<0$ can be

† To avoid explicit consideration of singularities of the kernel $\mathscr{L}$, one first performs the Fourier transform $t\to\zeta$ and then the limit $y, y'\to 0$.

added to the integration path. Hence, the real poles at ξ_l also contribute. Making use of $\mathrm{Res}[D'_\nu(-\zeta)/D_\nu(-\zeta)] = -1$ we obtain

$$\mathscr{L}(t) = -\mathrm{i}\sum_{l'} \mathrm{e}^{-\mathrm{i}\xi_{l'}t} \qquad (t \to +\infty). \tag{19}$$

To make the diffraction problem definite, we introduce, as always, a small dissipation, replacing ξ_l by $\xi_l - \mathrm{i}\sigma$, with $\sigma \to +0$. It means that in the kernel of (18) $D_\nu(-\zeta)$ is to be replaced by $D_\nu(-\zeta - \mathrm{i}\sigma)$, and that $f(t)$ is to be multiplied by $\mathrm{e}^{-\sigma t}$. After the dissipation σ is introduced, the integration path in kernel (18) can be chosen along the real axis ξ. Changing the integration variable in the second term, $\zeta \to -\zeta$, we get

$$\mathscr{L}(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}\zeta}{2\pi} \mathrm{e}^{\mathrm{i}\zeta t} H(\zeta) \tag{20}$$

where

$$H(\zeta) = \frac{D'_\nu(-\zeta)}{D_\nu(-\zeta - \mathrm{i}\sigma)} + \frac{D'_\nu(\zeta)}{D_\nu(\zeta - \mathrm{i}\sigma)} = -\frac{(2\pi)^{1/2}}{\Gamma(-\nu)} \frac{1}{D_\nu(\zeta - \mathrm{i}\sigma) D_\nu(-\zeta - \mathrm{i}\sigma)} = H(-\zeta). \tag{21}$$

Here the Wronskian (A18) was used, and we put $\sigma = 0$ in the functions D_ν entering in the numerator.

3. Solution of the integral equation

Integral equation (17) can be solved by the Wiener–Hopf technique [18]. To do this, we define a new unknown function for $t < 0$,

$$u(t) = \int_0^\infty \mathrm{d}t' \, v(t') \mathscr{L}(t - t') \tag{22}$$

and extend the functions v, f and u to the entire t axis, assuming that by definition $v = 0$ and $f = 0$ for $t < 0$, and $u = 0$ for $t > 0$. With these definitions one can see that the following equation is valid for all t values:

$$\int_{-\infty}^{+\infty} \mathrm{d}t' \, v(t') \mathscr{L}(t - t') = u(t) + f(t). \tag{23}$$

Perform the Fourier transformation

$$g(t) \to \bar{g}(\zeta) = \int_{-\infty}^{\infty} \mathrm{d}t \, \mathrm{e}^{-\zeta t} g(t). \tag{24}$$

The Fourier transform of the kernel $\mathscr{L}(t)$ is $H(\zeta)$. The singularities of $H(\zeta)$ are poles,

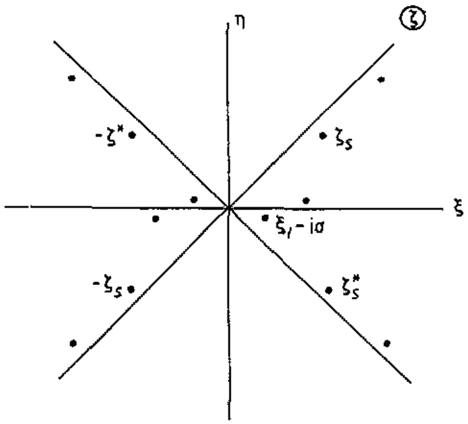

Figure 2. Location of the poles of the Fourier transform (21) of kernel (18).

the locations of which are shown in figure 2. The poles of the first term in (21) are the 'real' poles at $\xi_l - i\sigma$ and the complex poles at ζ_S. The residues of these poles are -1. The poles of the second term are at $-\xi_l + i\sigma$ and $-\zeta_S$ with residues $+1$. The complex poles of $H(\zeta)$ are located symmetrically with respect to the real and imaginary axes: $\pm\zeta_S$, $\pm\zeta_S^*$. Function $H(\zeta)$ is regular in the strip $|\eta| \leqslant \sigma'$, where $\sigma' = \sigma - 0$ (see figure 3).

The analytical properties of the transform $\bar{v}(\zeta)$ are determined by the properties of the field $\varphi(x)$. It follows from physical arguments that $\varphi(x)$ is exponentially small when $|x| \gg a_H$. Hence, $\bar{v}(\zeta)$ is obviously regular at $\eta < \sigma'$, i.e. in the strip and below it. The same is true for the transform

$$\bar{f}(\zeta) = i(\zeta + \xi_l - i\sigma)^{-1}. \tag{25}$$

Since for large t kernel $\mathcal{L}(t) \sim e^{-\sigma|t|}$ and since $v(t)$ is exponentially small for $t \gg 1$, one can see that $u(t) \sim e^{\sigma t}$ as $t \to -\infty$. Hence, $\bar{u}(\zeta)$ is regular at $\eta > -\sigma$, i.e. in the strip and above it.

As a result, the Fourier transforms of all the functions which enter (23) are regular in the strip $|\eta| \leqslant \sigma' < \sigma$. Hence, the following equation is valid in the strip:

$$\bar{v}(\zeta)H(\zeta) = \bar{u}(\zeta) + \bar{f}(\zeta) \qquad |\eta| \leqslant \sigma'. \tag{26}$$

The next step in the Wiener–Hopf technique is the factorization of the kernel,

$$H(\zeta) = H_+(\zeta)H_-(\zeta) \tag{27}$$

where $H_+(\zeta)$ is a function with all its singularities below the strip, while $H_-(\zeta)$ is a function with all its singularities above the strip. The calculation of these functions is given in the next section. The result is

$$H_+(\zeta) = [H(0)]^{1/2} \exp(\tfrac{1}{8}\zeta^2 - ic\zeta) \frac{1}{P_+(\zeta)Q_+(\zeta)} \tag{28}$$

where

$$P_+(\zeta) = \prod_l \left(1 - \frac{\zeta}{\xi_l - i\sigma}\right) \qquad Q_+(\zeta) = \prod_s^{(-)} \left(1 - \frac{\zeta}{\zeta_s}\right) e^{\zeta/\zeta_s}. \tag{29}$$

$P_+(\zeta)$ is a polynomial, the order of which is equal to the number of propagating edge states, and $Q_+(\zeta)$ is an infinite product over all the complex poles of $H(\zeta)$ located in the lower half-plane. It follows from the asymptotic behaviour of ζ_s, given by (A21), that $Q_+(\zeta)$ converges when combining the factors, which correspond to the poles ζ_s and $-\zeta_s^*$, located symmetrically with respect to the imaginary axis.

The real quantity c entering in (28) is given by the principal value of the integral

$$c = \frac{1}{2\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}\zeta}{\zeta} \frac{H'(\zeta)}{H(\zeta)}\bigg|_{\sigma=0}. \tag{30}$$

Factorization (27) is assumed to be symmetric, i.e.

$$H_-(\zeta) = H_+(-\zeta). \tag{31}$$

It follows from the symmetry of the location of the poles of $H(\zeta)$ that

$$H_-(\zeta) = [H(0)]^{1/2} \exp(\tfrac{1}{8}\zeta^2 + \mathrm{i}c\zeta) \frac{1}{P_-(\zeta)Q_-(\zeta)} \tag{32}$$

where

$$P_-(\zeta) = \prod_l \left(1 - \frac{\zeta}{\xi_l + \mathrm{i}\sigma}\right) = P_+(-\zeta)$$
$$Q_-(\zeta) = \prod_s^{(+)} \left(1 - \frac{\zeta}{\zeta_s}\right) \mathrm{e}^{\zeta/\zeta_s} = Q_+(-\zeta). \tag{33}$$

From (28) and (32) one can obtain $H(\zeta)$ as an infinite product

$$H(\zeta) = H(0) \exp(\tfrac{1}{4}\zeta^2)[P_+(\zeta)P_-(\zeta)Q_+(\zeta)Q_-(\zeta)]^{-1}. \tag{34}$$

Function $H_+(\zeta)$ has no zeros in the strip $|\eta| \leqslant \sigma'$, and hence (26) can be written in the form

$$\bar{v}(\zeta)H_-(\zeta) = \frac{\bar{u}(\zeta)}{H_+(\zeta)} + h(\zeta) \qquad h(\zeta) = \frac{\bar{f}(\zeta)}{H_+(\zeta)}. \tag{35}$$

Now we have to perform the decomposition of the excitation term of the integral equation,

$$h(\zeta) = h_+(\zeta) + h_-(\zeta)$$

where h_+ is regular in the strip and above it, and h_- is regular in the strip and below it. It is easy to verify that

$$h_-(\zeta) = \frac{\bar{f}(\zeta)}{H_+(-\xi_l + \mathrm{i}\sigma)} \qquad h_+(\zeta) = \bar{f}(\zeta)\left(\frac{1}{H_+(\zeta)} - \frac{1}{H_+(-\xi_l + \mathrm{i}\sigma)}\right). \tag{36}$$

Function $h_-(\zeta)$ differs from $\bar{f}(\zeta)$ by a constant factor, while the pole of $\bar{f}(\zeta)$ in the expression for $h_+(\zeta)$ is quenched by the zero of the factor in brackets.

Substituting decomposition (36) in (35) gives

$$\bar{v}(\zeta)H_-(\zeta) - h_-(\zeta) = \frac{\bar{u}(\zeta)}{H_+(\zeta)} + h_+(\zeta) \equiv p(\zeta). \tag{37}$$

If one performs the analytical continuation of both sides of (37) from the strip $|\eta| \leqslant \sigma'$ to the entire complex plane, the LHS becomes a function regular for $\eta \leqslant \sigma'$, while the RHS becomes a function regular for $\eta \geqslant -\sigma'$. Hence, $p(\zeta)$ is an entire function.

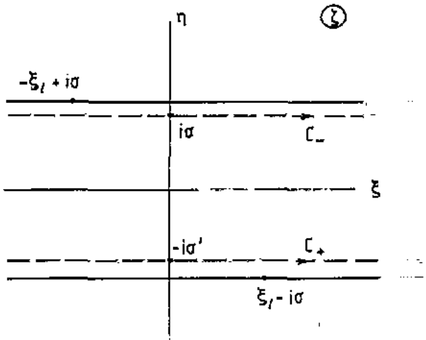

Figure 3. Strip $|\eta| < \sigma$, where all the Fourier transforms are regular, and the integration paths $C_\pm$.

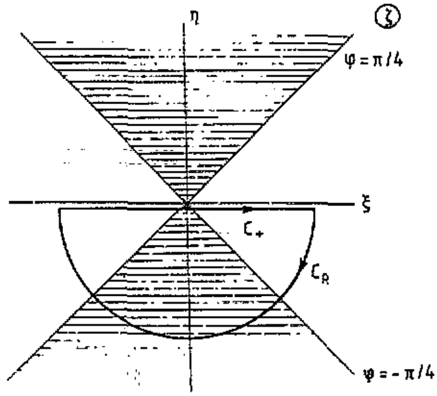

Figure 4. See text.

Actually, $p(\zeta) \equiv 0$, since $p(\zeta) \to 0$ as $\zeta \to \infty$. To prove this, consider the LHS of (37) as $\zeta \to \infty$ in the lower half-plane. It is obvious from (36) and (25) that $h_-(\zeta) \to 0$ as $\zeta \to \infty$. The behaviour of $\bar{v}(\zeta)$ as $\zeta \to \infty$ is determined by the behaviour of $v(t)$ as $t \to 0$. In [17] it was demonstrated that at small distances the wavefunction ψ in a magnetic field obeys the Laplace equation. Hence, the singularity of ψ as $x \to 0$ is the same as the singularity of the electrostatic potential near the edge of a metallic plane with a zero potential. As a result, $v(t) \sim t^{1/2}$ as $t \to 0$, and correspondingly $\bar{v}(\zeta) \sim \zeta^{-3/2}$ as $\zeta \to \infty$. Making use of (57), we can see now that $H_+(\zeta)\bar{v}(\zeta) \to 0$ as $\zeta \to \infty$ QED.

Making use of $p(\zeta) \equiv 0$, one finds eventually the Fourier transform of the solution of integral equation (17):

$$\bar{v}(\zeta) = h_-(\zeta)/H_-(\zeta). \tag{38}$$

4. Factorization of the kernel

To factorize the kernel $H(\zeta)$, we consider first its logarithmic derivative

$$\Lambda(\zeta) = \frac{d}{d\zeta} \ln H(\zeta) = \frac{H'(\zeta)}{H(\zeta)} \tag{39}$$

and decompose it [18]

$$\Lambda(\zeta) = \Lambda_+(\zeta) + \Lambda_-(\zeta) \tag{40}$$

where Λ_+ is regular for $\eta > -\sigma$, and Λ_- is regular for $\eta < \sigma$. In these domains the functions $\Lambda_\pm$ are given by integrals

$$\Lambda_\pm(\zeta) = \pm\frac{1}{2\pi i}\int_{C_\pm} d\zeta \frac{\Lambda(\zeta')}{\zeta' - \zeta}. \tag{41}$$

The integration path C_+ is from $\xi = -\infty$ to $\xi = +\infty$ along the lower boundary of the strip $|\eta| \leqslant \sigma'$, while C_- is in the same direction along the upper boundary (figure 3). One can easily verify that

$$\Lambda_-(\zeta) = -\Lambda_+(-\zeta). \tag{42}$$

The factorization of the kernel $H(\zeta)$ is given in terms of the function $\Lambda_\pm(\zeta)$ as follows:

$$H_\pm(\zeta) = \exp\left(\int_{\zeta_\pm}^{\zeta} \mathrm{d}\zeta' \, \Lambda_\pm(\zeta')\right) \tag{43}$$

where $\zeta_\pm$ are arbitrary integration constants. To ensure symmetry relation (31), we have to choose $\zeta_- = -\zeta_+$.

The asymptotic properties of $\Lambda(\zeta)$ as $\zeta = \rho\, \mathrm{e}^{\mathrm{i}\varphi} \to \infty$ can be found from the known asymptotic properties of the functions $D_\nu(\zeta)$ [19]. The result is as follows: in the lower and upper sectors (hatched in figure 4)

$$\Lambda(\zeta) = \zeta - 2\nu/\zeta + \dots \tag{44}$$

while in the right and left ones

$$\Lambda(\zeta) = 1/\zeta + 2(2\nu + 1)/\zeta^3. \tag{45}$$

Because of the growth of $\Lambda(\zeta)$ in the lower and upper sectors, one cannot close the integration path in (41). This is the reason for calculating first the function

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\Lambda_+(\zeta) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{C}_+} \mathrm{d}\zeta' \frac{\Lambda(\zeta')}{(\zeta' - \zeta)^2}. \tag{46}$$

Represent the integral along C_+ as an integral along the closed contour $\mathrm{C}_+ + \mathrm{C}_\mathrm{R}$ minus the integral along C_R. Here C_R is a semicircle in the lower half-plane, with radius $R \to \infty$ (see figure 4). For $\eta > -\sigma'$ the integral along $\mathrm{C}_+ + \mathrm{C}_\mathrm{R}$ is a sum of contributions from the poles ζ_s of $\Lambda(\zeta)$ in the lower half-plane. These poles are in the same points as the poles of $H(\zeta)$, and $\mathrm{Res}\,\Lambda(\zeta_r) = -1$. As a result

$$\lim_{R\to\infty} \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{C}_+ + \mathrm{C}_R} (\dots) = \sum_r^{(-)} \frac{1}{(\zeta - \zeta_r)^2}. \tag{47}$$

The sum over r runs over a finite number of 'real' poles $\xi_l - \mathrm{i}\sigma$ and over an infinite number of complex poles ζ_s and $-\zeta_s^*$. The infinite sum converges if one combines the terms corresponding to ζ_s and $-\zeta_s^*$.

Calculating the integral along C_R, we can neglect ζ compared to ζ' and replace $\Lambda(\zeta')$ by the asymptotic expansions (44) and (45). Performing the limit $R \to \infty$, one finds the integral to be equal to $-1/4$. Combining both integrals we have

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\Lambda_+(\zeta) = \frac{1}{4} + \sum_r^{(-)} \frac{1}{(\zeta - \zeta_r)^2}. \tag{48}$$

Integrating (48) we obtain

$$\Lambda_+(\zeta) = \Lambda_+(0) + \tfrac{1}{4}\zeta + \sum_r^{(-)} \frac{\zeta}{\zeta_r(\zeta_r - \zeta)} = \Lambda_+(0) + \tfrac{1}{4}\zeta - \frac{\mathrm{d}}{\mathrm{d}\zeta}\sum_r^{(-)} \ln\left[\left(1 - \frac{\zeta}{\zeta_r}\right)\mathrm{e}^{\zeta/\zeta_r}\right]. \tag{49}$$

Introducing (49) into (43) we have

$$H_+(\zeta) = H_+(0)\exp[\tfrac{1}{8}\zeta^2 + \Lambda_+(0)\zeta]\prod_r^{(-)}\left(1 - \frac{\zeta}{\zeta_r}\right)^{-1} \mathrm{e}^{-\zeta/\zeta_r}. \tag{50}$$

Setting $\zeta = 0$ in (27) we find

$$H_+(0) = H_-(0) = [H(0)]^{1/2}. \tag{51}$$

To calculate $\Lambda_+(0)$ from (41) we can choose the integration path along the real axis since $\Lambda(0) = 0$. After performing the limit $\sigma \to 0$, the integration path C_+ encompasses the poles ξ_l from above and the poles $-\xi_l$ from below (see figure 3). As a result, the integral is represented as a sum of two terms, the first being the sum of half-residues and the second being the principal part of the integral, i.e.

$$\Lambda_+(0) = \sum_l \frac{1}{\xi_l} - \mathrm{i}c \tag{52}$$

where c is given by (30). The first term in (52) when introduced into (50) cancels the exponential factors, corresponding to the real poles of the infinite product, and we obtain (28).

The asymptotic behaviour of infinite products like $Q_+(\zeta)$ in (29) for $\zeta \to \infty$ depends on the asymptotic behaviour of zeros ζ_s for $s \to \infty$ [18]. Since there is no simple function with an asymptotic behaviour of zeros like in (A21), we come back to the integral representation on (41) of function Λ_+. For $\eta > \sigma$ we can represent $\Lambda_+(\zeta)$ as an integral along the real axis and split the integral into a sum of two parts,

$$\Lambda_+(\zeta) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\xi'}{\xi' - \zeta} M(\xi') + \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi'}{\xi' - \zeta} \frac{\xi'}{\xi'^2 + 1} \tag{53}$$

with

$$M(\xi) = \Lambda(\xi) - \frac{\xi}{\xi^2 + 1} = -M(-\xi). \tag{54}$$

It follows from (45) that for $\xi \to \pm\infty$

$$M(\xi) = (4\nu + 3)\xi^{-3}. \tag{55}$$

In the first integral we expand $(\xi' - \zeta)^{-1}$ in powers of ξ'/ζ and find that this integral is of the order of ζ^{-2} as $\zeta \to \infty$. The second integral is equal to $\frac{1}{2}(\zeta + \mathrm{i})$. As a result

$$\Lambda_+(\zeta) = \tfrac{1}{2}\zeta + \ldots \qquad (\zeta \to \infty, \eta > 0). \tag{56}$$

Substituting (56) into (43) one finds

$$H_+(\zeta) \sim \zeta^{1/2} \qquad (\zeta \to \infty, \eta > 0). \tag{57}$$

5. Scattering matrix

First we calculate the scattered field ψ' in the far right side of the lower half-plane $(x = +\infty, y < 0)$. It follows from (7) that for $y < 0$

$$\psi'(x, y) = -\frac{1}{2m} \int_{-\infty}^{+\infty} \mathrm{d}x' \, \varphi(x') \int \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{\mathrm{i}k(x'-x)} \Phi_k'(0) \Psi_k(-y). \tag{58}$$

Introduce the non-dimensional variables t and v from section 2 and substitute Ψ_k and Φ_k' from (A17) and (15), respectively. Then we obtain

$$y < 0: \qquad \psi'(x, y) = m^{1/2} \int \frac{\mathrm{d}\zeta}{2\pi} \bar{v}(\zeta) \, \mathrm{e}^{\mathrm{i}kx} \frac{D_\nu(-s - \zeta)}{D_\nu(-\zeta)}. \tag{59}$$

As $x \to +\infty$, only the 'real' poles $\xi_l - \mathrm{i}\sigma$ contribute to the integral (see appendix).

Making use of (A11), one obtains

$$y<0, x\to+\infty: \qquad \psi'(x,y) = -\mathrm{i}\sum_{l'} \bar{v}(\xi_{l'} - \mathrm{i}\sigma)\bar{\Psi}_{El'}(x,y). \tag{60}$$

Comparing (60) with (1), using $\bar{v}(\zeta)$ from (38) with $h_-(\zeta)$ and $\bar{f}(\zeta)$ from (36) and (25), we find the scattering matrix

$$S_{ll'}(E) = \frac{1}{\xi_l + \xi_{l'}} \frac{1}{H_+(-\xi_l)H_+(-\xi_{l'})}. \tag{61}$$

Here we put $\sigma = 0$, since the points $\zeta = -\xi_l$ are far from the poles of $H_+(\zeta)$. The scattering matrix is symmetric in l and l'. This symmetry results from the time–magnetic field inversion combined with reflection from the xz plane.

The matrix elements $S_{ll'}$ are given in terms of a rather complicated function $H_+(\zeta)$, but the squares $|S_{ll'}|^2$ can be found in a very simple closed form. Consider first

$$|H_+(\xi)|^2 = H(0)\,\mathrm{e}^{\xi^2/4}|P_+(\xi)|^{-2}\,|Q_+(\xi)|^{-2} \tag{62}$$

for real argument ξ. From the symmetry of the complex poles it follows that

$$Q_+(\xi)^* = Q_-(\xi). \tag{63}$$

Making use of this relation and (33) and (34), we can eliminate the infinite product from (62). As a result,

$$|H_+(\xi)|^2 = H(\xi)\,P_-(\xi)/P_+(\xi)^*. \tag{64}$$

When setting $\xi = -\xi_l$ in (64), we can set $\sigma = 0$ in $P_+(\xi)$ and omit the complex-conjugation sign. Function $H(\xi)$ has a pole at $\xi = -\xi_l$ with residual equal to $+1$, while function $P_-(\xi)$ has a zero. Performing the limit $\xi \to -\xi_l$ we obtain

$$|H_+(-\xi_l)|^2 = -P'_+(\xi_l)/P_+(-\xi_l) \tag{65}$$

with

$$P_+(\xi) = \prod_l (1 - \xi/\xi_l). \tag{66}$$

Now from (61) and (65) one obtains a very simple closed expression

$$|S_{ll'}(E)|^2 = \frac{1}{(\xi_l + \xi_{l'})^2} \frac{P_+(-\xi_l)}{P'_+(\xi_l)} \frac{P_+(-\xi_{l'})}{P'_+(\xi_{l'})}. \tag{67}$$

The possibility to obtain closed expressions for the squares of the scattering matrix elements is perhaps a general property of some diffraction problems. A similar result is obtained in the case of electromagnetic wave diffraction at the open end of a semi-infinite waveguide [20].

In what follows we give explicit expressions for $|S_{ll'}|^2$. If the energy E is between the bulk Landau levels $l = 0$ and $l = 1$, only one edge state with $l = 0$ exists, and obviously $|S_{00}|^2 = 1$. If E is between the Landau levels $l = 1$ and $l = 2$, two edge states exist with

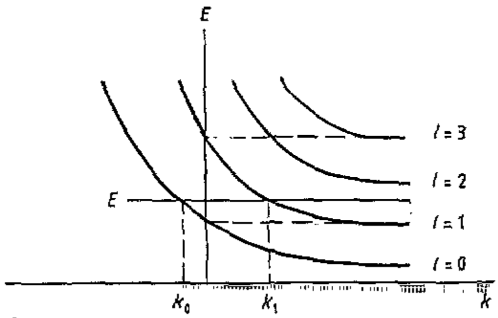

Figure 5. Plot of E versus k for edge states. Note that $E_l(+\infty) = \hbar\omega_H(l+\frac{1}{2})$ and $E_l(0) = \hbar\omega_H(2l+1+\frac{1}{2})$.

wavevectors $\xi_0 < 0 < \xi_1$ (see figure 5). From (67) it follows immediately that

$$|S_{00}|^2 = \frac{(\xi_0+\xi_1)^2}{(\xi_0-\xi_1)^2} = |S_{11}|^2$$
$$|S_{01}|^2 = -\frac{4\xi_0\xi_1}{(\xi_0-\xi_1)^2}. \tag{68}$$

When E is between the Landau levels $l = 2$ and $l = 3$, three edge states exist with $\xi_0 < 0 < \xi_1 < \xi_2$ and

$$|S_{00}|^2 = \frac{(\xi_0+\xi_1)^2(\xi_0+\xi_2)^2}{(\xi_0-\xi_1)^2(\xi_0-\xi_2)^2}$$
$$|S_{01}|^2 = -\frac{4\xi_0\xi_1(\xi_0+\xi_2)(\xi_1+\xi_2)}{(\xi_0-\xi_1)^2(\xi_0-\xi_2)(\xi_1-\xi_2)}. \tag{69}$$

The other matrix elements can be obtained by changing the indices. It is a straightforward calculation to verify flux conservation for (68) and (69). Note that (68) can be deduced from (69) by omitting all the factors involving ξ_2.

Of special interest is the case when energy E approaches the energy of a bulk Landau level $\hbar\omega_H(n+\frac{1}{2})$, i.e. $\nu = n + \delta, \delta \to 0$. When E is below the threshold ($\delta < 0$), equation (A8) has n roots, which are zeros of the Hermite polynomial $H_n(\xi/\sqrt{2})$. If n is even, the wavevectors ξ_l are divided into pairs $(\xi_0, \xi_{n-1} = -\xi_0)$, $(\xi_1, \xi_{n-2} = -\xi_1), \ldots$. If n is odd, except for the pairs there also exist a single mode $\xi_l = 0$ with $l = (n-1)/2$. When E is above the threshold ($\delta > 0$), an additional wavevector $\xi_n \gg 1$ appears. This wavevector corresponds to a quasi-bulk state located far from the boundary at $y_k = a_H\xi_n/\sqrt{2} \gg a_H$. For example, when $\nu \to 1$ one has $\xi_0 \to 0$ (single mode) and $\xi_1 \to +\infty$ (quasi-bulk mode). When $\nu \to 2$ one has $\xi_0 \to -\xi_1$ (pair of modes) and $\xi_2 \to +\infty$ (quasi-bulk mode). From (68) and (69) it follows that near the threshold ($\delta \to 0$) the single mode and the quasi-bulk modes are weakly mixed with other modes, while the pair modes are transformed nearly into each other.

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Appendix. Edge states and Green function

Here we collect for reference purposes some formulae related to the Schrödinger equation for a 2D electron confined to a half-plane in a magnetic field [21, 22].

We choose the Landau gauge of the vector potential

$$A_x = -Hy \qquad A_y = 0. \tag{A1}$$

In this case the edge states in the upper half-plane $y > 0$ are eigenfunctions of the Hamiltonian (2) satisfying the following boundary conditions: $\psi = 0$ at $y = 0$ and $y = +\infty$. The edge states are of the form

$$\psi(x, y) = N\,\mathrm{e}^{ikx}\chi(y) \tag{A2}$$

where k is the wavevector, and function χ obeys

$$\begin{aligned} &\mathcal{H}_k\chi = E\chi \qquad \chi(0) = \chi(+\infty) = 0 \\ &\mathcal{H}_k = -\frac{1}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2ma_{\mathrm{H}}^2}(y - y_k)^2 \qquad y_k = ka_{\mathrm{H}}^2,\ a_{\mathrm{H}}^2 = c/|e|H. \end{aligned} \tag{A3}$$

The equation for χ can be transformed to a non-dimensional one, if the following variables are introduced:

$$S = \sqrt{2}\,y/a_{\mathrm{H}} \qquad \xi = \sqrt{2}\,ka_{\mathrm{H}} \qquad \nu + \tfrac{1}{2} = E/\hbar\omega_{\mathrm{H}} \qquad \omega_{\mathrm{H}} = |e|H/mc.$$

The non-dimensional equation reads

$$\left(-\frac{\partial^2}{\partial S^2} + \tfrac{1}{4}(S - \xi)^2\right)\chi = (\nu + \tfrac{1}{2})\chi. \tag{A4}$$

Two linearly independent solutions are the parabolic cylinder functions [19]

$$D_\nu(S - \xi) \qquad D_\nu(-S + \xi). \tag{A5}$$

Since $\nu > 0$, we have

$$D_\nu(S) \to 0 \qquad D_\nu(-S) \to \infty \qquad (S \to +\infty). \tag{A6}$$

Hence, from the boundary condition at $y = +\infty$, it follows that

$$\chi(y) = cD_\nu(S - \xi). \tag{A7}$$

From the boundary condition at $y = 0$ one has

$$D_\nu(-\xi) = 0. \tag{A8}$$

This equation has a finite number of real roots ξ_l ($l = 0, 1, \ldots, [\nu]$, where $[\nu]$ is the integer part of ν), where l is the number of nodes of function $\chi(y)$, excluding $y = 0$. The label l is also the number of the bulk Landau level, from which the edge state is derived. The roots $\xi_l(\nu)$ determine the wavevectors $k_l(E)$ of the edge states present for given energy E (see figure 5).

From equation (A3) one can prove that for any eigenstate

$$\partial E/\partial k = -(a_{\mathrm{H}}^2/2m)\chi'(0)^2 \tag{A9}$$

if the normalization is conventional

$$\int_0^\infty \mathrm{d}y\, \chi(y)^2 = 1.$$

Making use of this relation, it is easy to find the edge states normalized to unit flux

$$\Psi_{El}(x, y) = \frac{(2m)^{1/2}}{a_{\mathrm{H}}} \mathrm{e}^{\mathrm{i}k_l x} \frac{\chi_{El}(y)}{\chi'_{El}(0)} = m^{1/2} \mathrm{e}^{\mathrm{i}k_l x} \frac{D_\nu(S - \xi_l)}{D'_\nu(-\xi_l)}. \tag{A10}$$

Note that the normal derivative at the boundary $y = 0$ of all these states is the same.

The edge states at $y < 0$ for the same gauge of the vector potential can be obtained from states (A10) by changing the signs of x and y, i.e.

$$\tilde{\Psi}_{El}(x, y) = -\Psi_{El}(-x, -y) = -m^{1/2} \mathrm{e}^{-\mathrm{i}k_l x} \frac{D_\nu(-S - \xi_l)}{D'_\nu(-\xi_l)}. \tag{A11}$$

The additional minus sign in front of Ψ is introduced to represent the change of the sign of the derivative.

The Green function $G(x, y; x', y')$ obeys (5) in xy and the following boundary conditions:

(i) $G = 0$ at $y = 0$, $y = +\infty$, $x - x' \to +\infty$; and
(ii) G is a superposition of edge states propagating from right to left at $x - x' \to -\infty$.

Owing to the translational invariance in x the Green function has the form

$$G(x, y; x', y') = \int \frac{\mathrm{d}k}{2\pi} \mathrm{e}^{\mathrm{i}k(x-x')} G_k(y, y'). \tag{A12}$$

Function G_k is the Green function corresponding to $\mathcal{H}_k$ (A3), i.e.

$$[\mathcal{H}_k(y) - E] G_k(y, y') = -\delta(y - y'). \tag{A13}$$

Hence,

$$G_k(y, y') = G_k(y', y) = \begin{cases} \Phi_k(y)\Psi_k(y') & y < y' \\ \Psi_k(y)\Phi_k(y') & y > y' \end{cases} \tag{A14}$$

where Φ_k and Ψ_k are solutions of equation (A3) with boundary conditions

$$\Phi_k(0) = 0 \qquad \Psi_k(+\infty) = 0 \tag{A15}$$

and normalized by the condition

$$W_y\{\Phi, \Psi\} = \Phi\Psi' - \Psi\Phi' = 2m \tag{A16}$$

where W_y is the Wronskian in y.

It follows now from (A4) and (A6) that

$$\begin{aligned} \Psi_k(y) &= AD_\nu(S-\xi) \\ \Phi_k(y) &= B[D_\nu(\xi)D_\nu(S-\xi) - D_\nu(-\xi)D_\nu(-S+\xi)]. \end{aligned} \tag{A17}$$

Making use of

$$W_S\{D_\nu(S), D_\nu(-S)\} = (2\pi)^{1/2}/\Gamma(-\nu) \tag{A18}$$

we find from (A16)

$$AB = (ma_{\rm H}/\pi^{1/2})\Gamma(-\nu)/D_\nu(-\xi). \tag{A19}$$

To satisfy the boundary conditions at $x - x' \to \pm\infty$, we choose the proper integration path in (A12). Consider G_k for complex k, i.e. change the real variable ξ to a complex one $\zeta = \xi + i\eta$. Since $D_\nu(\zeta)$ is an entire function in ζ, the only singularities of G_k in ζ are poles, which correspond to the zeros of the denominator in AB (A19). The zeros of the parabolic cylinder functions

$$D_\nu(-\zeta) = 0 \tag{A20}$$

define first-order poles. A finite number of these poles are on the real axis at points ξ_l. These real poles correspond to propagating edge states, see (A8). Besides, there exists an infinite number of complex poles ζ_S at complex-conjugate points. The complex poles correspond to evanescent edge states.

Using the asymptotic expressions for $D_\nu(-\zeta)$ as $\zeta \to \infty$, one can find that for large $|\zeta|$ the roots of (A20) are near the Stokes lines

$$\zeta_s = 2\pi^{1/2}\, e^{\pm i\pi/4} s^{1/2}. \tag{A21}$$

Bearing in mind the analytical properties of G_k in the ζ plane one can verify that the boundary conditions for G at $x - x' \to \pm\infty$ are satisfied if the integration path in (A12) encompasses all the real poles from above.

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